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# FIBONACCI DETERMINANTS – A COMBINATORIAL APPROACH

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## ABSTRACT

In this paper, we provide combinatorial interpretations for some determinantal identities involving Fibonacci numbers. We use the method due to Lindström-Gessel-Viennot in which we count nonintersecting  $n$ -routes in carefully chosen digraphs in order to gain insight into the nature of some well-known determinantal identities while allowing room to generalize and discover new ones.

## 1. INTRODUCTION

When a matrix has Fibonacci entries, combinatorial methods can bring deeper understanding to the evaluation of its determinant. While there are several combinatorial interpretations of determinants [6, 15], we choose to exploit the one due to Lindström-Gessel-Viennot [8, 12] that counts nonintersecting paths.

For an  $n \times n$  matrix  $A = \{a_{ij}\}$ , the general idea is to create an acyclic directed graph  $D$  with  $n$  origin nodes and  $n$  destination nodes, so that the number of paths from origin  $o_i$  to destination  $d_j$  is  $a_{ij}$ . For example, if  $A$  is the  $n \times n$  matrix of binomial coefficients  $A = \{\binom{i+j}{i}\}$ , where  $i$  and  $j$  range between 0 and  $n-1$ , then  $D$  can be constructed as follows:

1. the vertices of  $D$  are the integer points  $(i, j)$  where  $0 \leq i, j \leq n-1$ ;
2. the arcs of  $D$  create the grid on the integer lattice and are directed up or to the right;
3. origin  $o_i$  is the vertex  $(0, n-1-i)$  for  $0 \leq i \leq n-1$ ;
4. destination  $d_j$  is the vertex  $(j, n-1)$  for  $0 \leq j \leq n-1$ .

See Figure 1. To get from  $o_i$  to  $d_j$  takes  $j$  horizontal steps and  $i$  vertical steps, so there are  $\binom{i+j}{i}$  possible paths, as desired.

[width = 3.5in]figure01

Figure 1: The  $4 \times 4$  binomial matrix and its associated directed graph.

Origin nodes are enclosed by a circle and destination nodes are enclosed by a square.

An  $n$ -route in  $D$  is a collection of  $n$  directed paths, one from each origin, onto the set of destinations. Let  $X$  denote the set of indices of our origins and destinations, typically  $X = \{0, 1, \dots, n\}$  or  $X = \{1, 2, \dots, n\}$ , and let  $S_X$  denote the set of permutations of  $X$ . Then

every  $n$ -route induces a permutation  $\sigma$  in  $S_X$  where the  $i$ th directed path goes from origin  $o_i$  to destination  $d_{\sigma(i)}$ . For  $\sigma$  in  $S_X$ , the number of  $n$ -routes associated with  $\sigma$  is the product of the matrix entries corresponding to  $\sigma$  in  $A$ ,  $\prod_{i \in X} a_{i\sigma(i)}$ . Figure 2 shows a 4-route associated with the permutation  $(031)(2)$  for the matrix of binomial coefficients example. There are  $\binom{3}{0}\binom{1}{1}\binom{3}{2}\binom{4}{2} = 18$  4-routes corresponding to this permutation.

[width = 2.0in]figure02

Figure 2: One of the eighteen 4-routes associated with the permutation  $(031)(2)$ .

The permanent of  $A$  is the sum of over all permutations  $\sigma$  in  $S_X$  of  $\prod_{i \in X} a_{i\sigma(i)}$ , which counts the  $n$ -routes in the associated digraph  $D$ . By contrast, the determinant of  $A$ , denoted by  $\det(A)$  is equal to  $\sum_{\sigma \in S_X} \text{sgn}(\sigma) \prod_{i \in X} a_{i\sigma(i)}$ , where  $\text{sgn}(\sigma)$  is the sign of the permutation. So we interpret the determinant as the number of  $n$ -routes induced by even permutations minus the number of  $n$ -routes induced by odd permutations. As our next theorem will show, we can simplify the interpretation further by pairing some of the odd and even  $n$ -routes.

We call an  $n$ -route *intersecting* if there is a vertex of  $D$  shared by two of its paths and *nonintersecting* otherwise. For the 4-route given in Figure 2, the vertices of intersection are  $(0, 1)$ ,  $(0, 2)$ ,  $(0, 3)$ ,  $(1, 2)$ ,  $(1, 3)$ , and  $(2, 3)$ . For an intersecting  $n$ -route, *swapping the tails* of two of its paths at the “first point of intersection” (as described in [1]) introduces a transposition and transforms an even  $n$ -route to an odd  $n$ -route. Thus, by tailswapping, there is a bijection between even intersecting  $n$ -routes and odd intersecting  $n$ -routes. This yields the following theorem.

**Theorem 1:** *Let  $D$  be a directed acyclic graph with  $n$  designated origin and destination nodes, and let  $A$  be the  $n \times n$  matrix whose  $(i, j)$  entry is the number of paths from the  $i$ th origin to the  $j$ th destination. Then the determinant of  $A$  equals  $\text{Even}(D) - \text{Odd}(D)$ , where  $\text{Even}(D)$  is the number of nonintersecting  $n$ -routes corresponding to even permutations, and  $\text{Odd}(D)$  is the number of nonintersecting  $n$ -routes corresponding to odd permutations.*

Revisiting the example of  $A = \{\binom{i+j}{i}\}$  and its associated digraph  $D$ , there is only one nonintersecting  $n$ -route and it corresponds to the identity permutation (see Figure 3). Since the identity permutation is even,  $\det(A) = 1$ .

[width = 2.0in]figure03

Figure 3: The only nonintersecting 4-route for the digraph given in Figure 1 is associated with the identity permutation. Thus the determinant equals 1.

In fact, we can generalize the binomial determinant by shifting our destination nodes  $k$  units to the right. Then  $\det\{\binom{i+j+k}{i}\} = 1$  since there remains one nonintersecting  $n$ -route on the associated digraph.

In this paper we construct directed graphs that are used to calculate the determinants of matrices involving Fibonacci numbers and generalized Fibonacci numbers. By considering the signed sum of nonintersecting  $n$ -routes, we hope to concretely illustrate the beauty and elegance of these determinantal identities.

## 2. ELEMENTARY FIBONACCI DETERMINANTS

Determinants of matrices with Fibonacci entries have a long history [4, 9, 10]. For  $m \geq 1$ , Cassini’s identity

$$F_{m-1}F_{m+1} - F_m^2 = (-1)^m \quad (1)$$

is classically proved by showing that

$$\begin{bmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^m$$

and taking the determinant of both sides. We instead choose to evaluate the determinant on the left directly by counting nonintersecting 2-routes on a related directed graph. We define the *Fibonacci digraph*,  $\mathcal{F}_t$  on vertices  $v_0, v_1, v_2, \dots, v_t$  with arcs of the form  $(v_j, v_{j+1})$  (for  $0 \leq j \leq t-1$ ) called *steps*, and arcs of the form  $(v_j, v_{j+2})$  (for  $0 \leq j \leq t-2$ ) called *jumps*. The Fibonacci digraph  $\mathcal{F}_7$  is illustrated in Figure 4.

Figure 4: The directed graph  $\mathcal{F}_7$  associated with Cassini's Fibonacci determinant,  $F_{m-1}F_{m+1} - F_m^2 = (-1)^m$ , when  $m = 7$ . The black arcs are required in the nonintersecting 2-route.

It is easy to prove by induction that the number of paths from  $v_0$  to  $v_j$  is  $F_{j+1}$ . Equivalently, we shall make repeated use of the following: *In the Fibonacci digraph  $\mathcal{F}_t$ , for all  $0 \leq i \leq j \leq t$ , the number of paths from  $v_i$  to  $v_j$  is  $F_{j-i+1}$ .* Thus, to combinatorially interpret the matrix

$$A = \begin{bmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{bmatrix},$$

we consider the graph  $\mathcal{F}_m$ , and designate the origin nodes to be  $o_1 = v_1, o_2 = v_0$ , and the destination nodes to be  $d_1 = v_{m-1}, d_2 = v_m$ . Hence, for  $1 \leq i, j \leq 2$ , the number of paths from  $o_i$  to  $d_j$  is  $a_{ij}$ . For example, the number of paths from  $o_1$  to  $d_1$  and  $d_2$  are  $F_{m-1}$  and  $F_m$ , respectively.

So by Theorem 1, the determinant of  $A$  can be calculated by considering the nonintersecting 2-routes in  $\mathcal{F}_m$ . The only way a 2-route can be nonintersecting is if it consists entirely of jumps. If  $m$  is even, the nonintersecting 2-route will be associated with the identity permutation (since  $o_1$  goes to  $d_1$  and  $o_2$  goes to  $d_2$ ), giving a determinant of 1. If  $m$  is odd, the nonintersecting 2-route will be associated with the transposition (12) (since  $o_1$  goes to  $d_2$  and  $o_2$  goes to  $d_1$ ), giving a determinant of  $-1$ . So Cassini's identity given in equation (1) holds.

By allowing the origin and destination nodes to be nonconsecutive, we obtain a generalized form of Cassini's identity. Consider positive integers,  $m, r, s$  where  $r \leq m$ . Then the matrix

$$A = \begin{bmatrix} F_{m-r} & F_{m+s-r} \\ F_m & F_{m+s} \end{bmatrix}$$

can be shown to have determinant  $(-1)^{m-1-r} F_r F_s$ , as follows. In the Fibonacci digraph  $\mathcal{F}_{m+s-1}$ , we use origin nodes  $o_1 = v_r, o_2 = v_0$  and destination nodes  $d_1 = v_{m-1}$  and  $d_2 = v_{m+s-1}$ . Hence, for every  $1 \leq i, j \leq 2$ ,  $a_{ij}$  counts the paths from  $o_i$  to  $d_j$ . Thus by Theorem 1,  $\det(A)$  can be calculated by considering the nonintersecting 2-routes in  $\mathcal{F}_{m+s-1}$ . A nonintersecting 2-route contains some path from  $v_0$  to  $v_{r-1}$ , the jumps  $(v_i, v_{i+2})$  for  $r-1 \leq i \leq m-2$ , and some path from  $v_m$  to  $v_{m+s-1}$ . There are  $F_r F_s$  such 2-routes, and the parity of  $m-r$  determines which permutation is associated with the 2-route; that is,  $o_1$  goes to  $d_1$  in our nonintersecting 2-route if and only if  $m-1-r$  is even. So

$$F_{m-r}F_{m+s} - F_mF_{m+s-r} = \det(A) = (-1)^{m-1-r} F_r F_s. \quad (2)$$

[width = 4.0in]figure05

Figure 5: The directed graph associated with Cassini's generalized Fibonacci determinant,  $F_{m-r}F_{m+s} - F_mF_{m+s-r} = (-1)^{m+1-r}F_rF_s$ . The black arcs are required in a nonintersecting 2-route.

Increasing the number of origin and destination nodes under consideration in a Fibonacci digraph changes the size of the associated matrix. For example the determinant,

$$\det \begin{bmatrix} F_m & F_{m+1} & F_{m+2} \\ F_{m+1} & F_{m+2} & F_{m+3} \\ F_{m+2} & F_{m+3} & F_{m+4} \end{bmatrix}, \quad (3)$$

is associated with the nonintersecting 3-routes of  $\mathcal{F}_{m+3}$  with origins  $v_2, v_1, v_0$  and destinations  $v_{m+1}, v_{m+2}, v_{m+3}$ . Algebraically, the determinant is equal to zero because the third row is the sum of the first two. Combinatorially, the determinant is equal to zero because it is impossible to create a nonintersecting 3-route on  $\mathcal{F}_{m+3}$ .

By spreading out the locations of the origin and destination nodes, we see that a more general  $3 \times 3$  Fibonacci matrix has determinant equal to zero. Specifically if  $0 \leq r \leq s$  and  $0 \leq p \leq q$ , the digraph  $\mathcal{F}_{m+q+s-1}$  with origins  $o_1 = v_s, o_2 = v_{s-r}, o_3 = v_0$  and destinations  $d_1 = v_{m+s-1}, d_2 = v_{m+p+s-1}, d_3 = v_{m+q+s-1}$  has no nonintersecting 3-routes. See Figure 6. So

$$\det \begin{bmatrix} F_m & F_{m+p} & F_{m+q} \\ F_{m+r} & F_{m+p+r} & F_{m+q+r} \\ F_{m+s} & F_{m+p+s} & F_{m+q+s} \end{bmatrix} = 0, \quad (4)$$

which is not as obvious from the algebraic viewpoint. Similarly, Fibonacci matrices of higher order created in the same way must also have determinants equal to zero.

[width = 4.0in]figure06

Figure 6: There are no nonintersecting 3-routes on a Fibonacci digraph.

### 3. GENERALIZING FIBONACCI DETERMINANTS

#### The Gibonacci Digraph

A shrewd mathematician knows that lurking beneath any Fibonacci identity are related identities involving *Gibonacci* numbers. Next we modify the Fibonacci digraph  $\mathcal{F}_t$  to obtain some generalizations.

Recall that a Gibonacci sequence  $G_0, G_1, \dots, G_t, \dots$  is defined by initial conditions  $G_0, G_1$  and the recurrence  $G_t = G_{t-1} + G_{t-2}$  for  $t \geq 2$ . (When the initial conditions are  $G_0 = 0$  and  $G_1 = 1$ , these are the Fibonacci numbers.) It is easy to see, by induction on  $t$ , that for all  $t \geq 1$ ,

$$G_t = G_1 F_t + G_0 F_{t-1} \quad (5)$$

and

$$G_{r+t} = G_{r+1} F_t + G_r F_{t-1}. \quad (6)$$

For simplicity, we assume that  $G_0$  and  $G_1$  are positive integers, but we shall remove that assumption later. Now let  $S$  be a subset of  $\{0, 1, \dots, t-2\}$ . The *Gibonacci digraph*,  $\mathcal{G}_t(S)$ ,

includes a copy of  $\mathcal{F}_t$  on vertices  $v_0, v_1, v_2, \dots, v_t$ . Additionally, for each element  $i$  in  $S$ , we include a vertex  $v'_i$ , along with  $G_1$  copies of the arc  $(v'_i, v_{i+1})$  and  $G_0$  copies of the arc  $(v'_i, v_{i+2})$ . See Figure 7. In all of our applications,  $S$  will contain node 0, so that  $v'_0$  will be a vertex of our digraph. By considering the first step from  $v'_0$ , the number of paths from  $v'_0$  to  $v_s$  is  $G_1 F_s + G_0 F_{s-1} = G_s$  by equation (5). (Notice that when  $G_0 = 0$  and  $G_1 = 1$ , there are no arcs from  $v'_0$  to  $v_2$ , and so this amounts to the number of paths from  $v_1$  to  $v_s$  in the Fibonacci digraph, namely  $F_s$ .) In general, for  $i$  in  $S$  and  $t > i$ , the number of paths from  $v'_i$  to  $v_t$  is  $G_{t-i}$ .

[width = 4.0in]figure07

Figure 7: The Gibonacci digraph  $\mathcal{G}_7(\{0, 1\})$  associated with Cassini's Gibonacci identity.

For the Gibonacci version of Cassini's identity, we calculate the determinant of

$$A = \begin{bmatrix} G_{m-1} & G_m \\ G_m & G_{m+1} \end{bmatrix},$$

by considering nonintersecting 2-routes in  $\mathcal{G}_{m+1}(\{0, 1\})$  with origins  $o_1 = v'_1$ ,  $o_2 = v'_0$  and destinations  $d_1 = v_m$ ,  $d_2 = v_{m+1}$ . A nonintersecting 2-route contains the arcs  $(v_i, v_{i+2})$  for  $2 \leq i \leq m-1$ . See Figure 8.

[width = 4.0in]figure08

Figure 8: The directed graph associated with Cassini's generalized Gibonacci determinant,

$$G_{m-1}G_{m+1} - G_m^2 = (-1)^{m-1}(G_0G_2 - G_1^2).$$

The black arcs are required in a nonintersecting 2-route.

So it remains to determine how origins  $v'_1$  and  $v'_0$  connect to the rest of the 2-route through the vertices  $v_2$  and  $v_3$  in a nonintersecting way. There are  $G_0$  ways for  $v'_1$  to be incident to  $v_3$  and  $G_2$  paths from  $v'_0$  to  $v_2$ . Hence there are  $G_0G_2$  nonintersecting 2-routes that use an arc  $(v'_1, v_3)$ . The parity of this 2-route is determined by the parity of  $m$ , since  $o_1$  goes to  $d_1$  if and only if  $m$  is odd; hence all of these 2-routes have parity  $(-1)^{m-1}$ . Likewise, there are  $G_1$  ways for  $v'_1$  to be incident to  $v_2$  and  $G_1$  paths from  $v'_0$  to  $v_3$  that can complete a nonintersecting 2-route. Thus there are  $G_1^2$  nonintersecting 2-routes of this type, all of which have parity  $(-1)^m$ . So the determinant becomes  $(-1)^{m-1}G_0G_2 + (-1)^mG_1^2$  giving the identity

$$G_{m-1}G_{m+1} - G_m^2 = \det(A) = (-1)^{m-1}(G_0G_2 - G_1^2). \quad (7)$$

We can again obtain a more general theorem, by spreading out the origins and destinations. For positive integers  $m, r, s$  with  $r \leq m$ , we evaluate the determinant of

$$A = \begin{bmatrix} G_{m-r} & G_{m+s-r} \\ G_m & G_{m+s} \end{bmatrix}$$

by letting  $o_1 = v'_r$ ,  $o_2 = v'_0$ ,  $d_1 = v_m$ , and  $d_2 = v_{m+s}$  in  $\mathcal{G}_{m+s}(\{0, r\})$ . A nonintersecting 2-route in this digraph contains the arcs  $(v_i, v_{i+2})$  for  $r+1 \leq i \leq m-1$  and one of  $F_s$  paths between  $v_{m+1}$  and  $v_{m+s}$ . There are  $G_0$  ways for  $v'_r$  to be incident to  $v_{r+2}$  and  $G_{r+1}$  paths from  $v'_0$  to  $v_{r+1}$ . The parity of  $m-r$  determines which permutation is associated with these 2-routes. There are  $G_1$  ways for  $v'_r$  to be incident to  $v_{r+1}$  and  $G_r$  paths from  $v'_0$  to  $v_r$  that

can be used to complete a nonintersecting 2-route. These 2-routes have opposite sign. So the determinant gives the identity

$$\begin{aligned} G_{m-r}G_{m+s} - G_mG_{m+s-r} &= \det(A) = (-1)^{m-r}(G_0G_{r+1} - G_rG_1)F_s \\ &= (-1)^{m-r}(G_0G_{r+s} - G_rG_s) \end{aligned}$$

where the second equality can be shown using equations (5) and (6). In fact, this argument can be generalized one step further. Let  $H_t$  be another Gibonacci sequence with initial conditions  $H_0$  and  $H_1$ . Modify the digraph  $\mathcal{G}_{m+s}(\{0, r\})$  by assigning  $H_1$  arcs from  $v'_0$  to  $v_1$  and  $H_0$  arcs from  $v'_0$  to  $v_2$ . Then virtually the same argument leads to

$$\begin{aligned} \det \begin{bmatrix} G_{m-r} & G_{m+s-r} \\ H_m & H_{m+s} \end{bmatrix} &= (-1)^{m-r}(G_0H_{r+1} - G_rH_1)F_s \\ &= (-1)^{m-r}(G_0H_{r+s} - G_rH_s). \end{aligned}$$

As in the case of the general  $3 \times 3$  Fibonacci determinant given in (4), it is impossible to create nonintersecting  $n$ -routes for  $n \geq 3$  in a Gibonacci digraph, provided the origin nodes precede the destination nodes. Consequently,

$$\det \begin{bmatrix} G_m & G_{m+p} & G_{m+q} \\ G_{m+r} & G_{m+p+r} & G_{m+q+r} \\ G_{m+s} & G_{m+p+s} & G_{m+q+s} \end{bmatrix} = 0, \quad (8)$$

or more generally, for Gibonacci sequences  $G_t$ ,  $H_t$ , and  $I_t$ ,

$$\det \begin{bmatrix} G_m & G_{m+p} & G_{m+q} \\ H_{m+r} & H_{m+p+r} & H_{m+q+r} \\ I_{m+s} & I_{m+p+s} & I_{m+q+s} \end{bmatrix} = 0. \quad (9)$$

Similarly, Gibonacci matrices of higher order created in the same way must also have determinants equal to zero.

### Combining Binomial and Fibonacci Digraphs

Now that we understand nonintersecting routes in the binomial digraph from Figure 1 and the Fibonacci digraph,  $\mathcal{F}_m$ , it is reasonable to investigate what happens when we combine them. For example, we can attach the Fibonacci digraph  $\mathcal{F}_m$  to the top of a binomial lattice to create  $\mathcal{H}_m$  as illustrated in Figure 9.

[width = 4.0in]figure09

Figure 9: The digraph  $\mathcal{H}_m$  contains the  $2 \times m$  binomial digraph plus the arcs  $((i, 1), (i + 2, 1))$  for  $0 \leq i \leq m - 2$ .

Assign origins  $o_1 = (0, 1)$ ,  $o_2 = (0, 0)$ , and destinations  $d_1 = (m - 1, 1)$ ,  $d_2 = (m, 1)$ . The number of paths from  $o_1$  to  $d_1$  and  $d_2$  are  $F_m$  and  $F_{m+1}$  respectively. Any path from  $o_2$  to the destinations must use exactly one vertical arc, which we interpret as the first jump of the path. Since jumps have length 2, the number of paths from  $o_2$  to  $d_1$  equals the number of

paths of length  $m+1$  that use at least one jump. Thus the number of paths from  $o_2$  to  $d_1$  and  $d_2$  are  $F_{m+2} - 1$  and  $F_{m+3} - 1$ , respectively. So we are ready to compute the determinant of

$$A = \begin{bmatrix} F_m & F_{m+1} \\ F_{m+2} - 1 & F_{m+3} - 1 \end{bmatrix}. \quad (10)$$

A nonintersecting 2-route in  $\mathcal{H}_m$  uses one arc for the form  $((q, 0), (q, 1))$ ,  $1 \leq q \leq m$ . There are  $F_q$  paths from  $(0, 1)$  to  $(q-1, 1)$  and the rest of the nonintersecting 2-route is uniquely determined. If  $m$  and  $q$  have the same parity, the associated permutation is even and the contribution to the determinant is  $+F_q$ . If  $m$  and  $q$  have opposite parity, the contribution is  $-F_q$ . So  $Even(\mathcal{H}_m) - Odd(\mathcal{H}_m) = \sum_{q=1}^m (-1)^{m-q} F_q = (-1)^{m-1} + F_{m-1}$ . The last equality comes from the familiar alternating Fibonacci sum identity

$$\sum_{q=1}^m (-1)^q F_q = -1 + (-1)^m F_{m-1}, \quad (11)$$

which can be proved combinatorially (as done in [3]) or by induction on  $m$ . Therefore,  $\det(A) = (-1)^{m-1} + F_{m-1}$ .

It is natural to generalize the graph. Begin by increasing the lattice from  $(0, 0)$  to  $(m, 2)$  and define the vertices  $o_1 = (0, 2)$ ,  $o_2 = (0, 0)$ ,  $d_1 = (m-1, 2)$ , and  $d_2 = (m, 2)$ . This digraph corresponds to the matrix

$$A = \begin{bmatrix} F_m & F_{m+1} \\ F_{m+4} - (m+3) & F_{m+5} - (m+4) \end{bmatrix}$$

since any path from  $o_2$  to the destinations must use at least two jumps. A nonintersecting 2-route uses one arc for the form  $((q, 1), (q, 2))$ ,  $1 \leq q \leq m$ . Given  $q$ , there are  $F_q$  paths from  $(0, 2)$  to  $(q-1, 2)$ , there are  $q+1$  choices for the first vertical arc, and the rest of the nonintersecting 2-route is uniquely determined. If  $m$  and  $q$  have the same parity, the associated permutation is even and the contribution to the determinant is  $+(q+1)F_q$ . If  $m$  and  $q$  have opposite parity, the contribution is  $-(q+1)F_q$ . So the determinant of  $A$  is  $\sum_{q=1}^m (-1)^{m-q} (q+1)F_q$ . Using the identity

$$\sum_{q=1}^m qF_q (-1)^q = (-1)^m (mF_{m-1} + F_{m-3}) - 2 \quad (12)$$

which can also be proved combinatorially or by induction on  $m$ , we can derive the determinant of  $A$  to be  $(m+1)F_{m-1} + F_{m-3} - 3 \cdot (-1)^m$ .

Now consider the same graph with three origins  $o_1 = (0, 2)$ ,  $o_2 = (0, 1)$ ,  $o_3 = (0, 0)$ , and three destinations  $d_1 = (m-2, 2)$ ,  $d_2 = (m-1, 2)$ , and  $d_3 = (m, 2)$ .   
 [width = 4.0in]figure10

Figure 10: The  $3 \times m$  binomial digraph plus the arcs  $((i, 2), (i+2, 2))$  for  $0 \leq i \leq m-2$ .

This digraph corresponds to the determinant of

$$A = \begin{bmatrix} F_{m-1} & F_m & F_{m+1} \\ F_{m+1} - 1 & F_{m+2} - 1 & F_{m+3} - 1 \\ F_{m+3} - (m+2) & F_{m+4} - (m+3) & F_{m+5} - (m+4) \end{bmatrix}.$$



In a nonintersecting 3-route,  $o_3$  must map to  $d_3$  using the arc  $((m, 1), (m, 2))$ . Then the path from  $o_2$  must use an arc  $((q, 1), (q, 2))$ , for  $1 \leq q \leq m-1$ . Given  $q$ , the path from  $o_3$  must use an arc  $((r, 0), (r, 1))$ , for  $q+1 \leq r \leq m$  and there are  $F_q$  paths from  $o_1$  to  $(q-1, 2)$ . If  $m$  and  $q$  have opposite parity, the associated permutation is even and the contribution to the determinant is  $+(m-q)F_q$ . If  $m$  and  $q$  have the same parity, the contribution is  $-(m-q)F_q$ . Counting the nonintersecting 3-routes and using equations (11) and (12), we find that

$$\det(A) = \sum_{q=1}^{m-1} (-1)^{m+1-q} (m-q)F_q = F_{m-3} + (-1)^m (m-2).$$

Clearly there is more work to be done increasing the height of the lattice and the distance between destinations.

### Weighting the Arcs

Suppose we take the matrix given in (3) and add a fixed constant  $k$  to each entry,

$$\begin{bmatrix} F_m + k & F_{m+1} + k & F_{m+2} + k \\ F_{m+1} + k & F_{m+2} + k & F_{m+3} + k \\ F_{m+2} + k & F_{m+3} + k & F_{m+4} + k \end{bmatrix}.$$

By adding weighted arcs to a directed graph, we will be able to calculate this determinant even when  $k$  is not an integer. In the digraph given in Figure 11, all arcs have weight 1, except for arc  $(o_1, w)$  of weight  $k$  and arc  $(w, d_3)$  of weight  $-1$ .  
[width = 4.0in] figure11

Figure 11: Adding vertices and weighted arcs to the Fibonacci digraph changes the determinant.

The *weight* of an  $n$ -route is the product of the weights of the arcs used. The 3-route in Figure 11 has weight  $-k$ . Switching destinations at a point of intersection preserves the weight of an  $n$ -route but changes the sign of the associated permutation. So just as in the unweighted situation, the determinant will be the sum of the weights of the even nonintersecting  $n$ -routes minus the sum of the weights of the odd nonintersecting  $n$ -routes.

Figure 11 is the right picture to verify that

$$\det(A) = \det \begin{bmatrix} F_m + k & F_{m+1} + k & F_{m+2} + k \\ F_{m+1} + k & F_{m+2} + k & F_{m+3} + k \\ F_{m+2} + k & F_{m+3} + k & F_{m+4} + k \end{bmatrix} = (-1)^{m+1} k.$$

To see that the matrix entries count weighted paths from origins to destinations, we verify total weights for the paths starting at the first and second origins. From  $o_1$  to  $d_1$ , there are  $F_m$  paths of weight 1 and one path of weight  $k$ , for a total weight of  $F_m + k$ . From  $o_1$  to  $d_2$  there are  $F_{m+1}$  paths of weight 1 and one path of weight  $k$ . From  $o_1$  to  $d_3$  there are  $F_{m+2}$  paths of weight 1, *two* paths of weight  $k$ , and one path of weight  $-k$ , for a total weight of  $F_{m+2} + k$ . Starting from  $o_2$  with an initial horizontal step, the total weight of paths to  $d_1$  is, as before,  $F_m + k$ , and the number of paths after an initial downward step to  $v_3$  is  $F_{m-1}$ ; hence the total weight of all paths from  $o_3$  to  $d_1$  is  $F_m + k + F_{m-1} = F_{m+1} + k$ . The other calculations are derived in a similar way.

The unique nonintersecting 3-route sends  $o_1$  to  $d_3$  via the path of weight  $-k$ . Origins  $o_2$  and  $o_3$  “jump” to  $d_1$  and  $d_2$  by paths of weight 1. If  $m$  is odd, then  $o_2$  goes to  $d_2$  and the

resulting permutation, in cycle notation (13), is odd. Otherwise the permutation is (132), which is even. Hence the determinant of  $A$  is  $(-1)^m(-k) = (-1)^{m+1}k$ , as desired.

This identity easily generalizes to Gibonacci entries, even with noninteger initial conditions. The digraph given in Figure 12 can be used to verify

$$\det \begin{bmatrix} G_m + k & G_{m+1} + k & G_{m+2} + k \\ G_{m+1} + k & G_{m+2} + k & G_{m+3} + k \\ G_{m+2} + k & G_{m+3} + k & G_{m+4} + k \end{bmatrix} = (-1)^m k (G_0 G_2 - G_1^2). \quad (13)$$

Figure 12: Adding weighted arcs to the Gibonacci digraph. The black arcs are required in a nonintersecting 3-route.

As before, a nonintersecting 3-route sends  $o_1$  to  $d_3$  via the path of weight  $-k$ . After using the initial weighted arcs, the paths from  $o_2$  and  $o_3$  to destinations  $d_1$  and  $d_2$  are completely determined. If the path from  $o_2$  has weight  $G_1$ , then the path from  $o_3$  must also have weight  $G_1$ . If the path from  $o_2$  has weight  $G_0$ , then the path from  $o_3$  has weight  $G_0$  or  $G_1$  (and thus the weight of all such paths is  $G_0 + G_1 = G_2$ ), but the final destinations will be transposed. The sign of the associated permutation is determined by the parity of  $m$ . Hence, the signed sum of the weighted 3-routes is  $(-1)^m k (G_0 G_2 - G_1^2)$ , as promised.

We note that this technique can be applied to our earlier determinant identities with Gibonacci numbers. By replacing the  $G_0$  or  $G_1$  arcs of weight 1 in Figures 7 and 8 with a single arc of weight  $G_0$  or  $G_1$ , the earlier identities remain true with essentially the same argument, even when the initial conditions are not positive integers.

The directed graph in Figure 12 is not the most intuitive representation for the determinant in equation (13). However, it has the advantage of a simple analysis of nonintersecting 3-routes. A more intuitive representation is the digraph in Figure 13 which takes  $\mathcal{G}_{m+4}(\{0, 1, 2\})$  and adds the vertices  $x, w$ , the arcs  $(v'_2, x)$ ,  $(v'_1, x)$ ,  $(v'_0, x)$ ,  $(w, v_{m+2})$  of weight one, the arc  $(x, w)$  of weight  $k$ , and the arc  $(w, v_{m+4})$  of weight  $-1$ . The analysis of nonintersecting 3-routes must now consider which of the three origin vertices gets mapped to  $d_3$  via the path of weight  $-k$ . The advantage here is that we can generalize equation (13) by introducing different weights at each origin. See Figure 13. Let  $H_t$  and  $I_t$  be Gibonacci sequences with initial conditions  $H_0, H_1$ , and  $I_0, I_1$  respectively. We evaluate the determinant of

$$A = \begin{bmatrix} G_m + k & G_{m+1} + k & G_{m+2} + k \\ H_{m+1} + k & H_{m+2} + k & H_{m+3} + k \\ I_{m+2} + k & I_{m+3} + k & I_{m+4} + k \end{bmatrix} \quad (14)$$

by counting nonintersecting 3-routes in Figure 13.

[width = 4.0in]figure13

Figure 13: A more intuitive and more general digraph than Figure 12. The black arcs are required in all nonintersecting 3-routes.

Suppose a nonintersecting 3-route sends  $o_1$  to  $d_3$  via the path of weight  $-k$ . After using the initial weighted arcs, the paths from  $o_2$  and  $o_3$  to destinations  $d_1$  and  $d_2$  are completely determined. If the path from  $o_2$  has weight  $H_1$ , then the path from  $o_3$  has weight  $I_1$ . If the path from  $o_2$  has weight  $H_0$ , then the path from  $o_3$  has weight  $I_0$  or  $I_1$  (and thus the weight of all such paths is  $I_2$ ), but the final destinations will be transposed. The sign of the associated permutation is determined by the parity of  $m$ . Hence, the contribution towards the signed sum of the weighted 3-routes is  $(-1)^m k (H_0 I_2 - H_1 I_1)$ . A similar analysis shows a contribution

of  $(-1)^m k(G_0 I_3 - G_1 I_2)$  when  $o_2$  maps to  $d_3$  via the path of weight  $-k$  and a contribution of  $(-1)^{m+1} k(G_0 H_2 - G_1 H_1)$  when  $o_3$  maps to  $d_3$  via the path of weight  $-k$ . Thus

$$\det(A) = (-1)^m k[(H_0 I_2 - H_1 I_1) + (G_0 I_3 - G_1 I_2) - (G_0 H_2 - G_1 H_1)]. \quad (15)$$

Since  $G_0 G_3 - G_1 G_2 = G_0 G_2 - G_1^2$ , the determinant in (15) specializes to (13) when the Gibonacci sequences coincide. Further, selecting initial conditions  $H_0 = G_{r-1}$ ,  $H_1 = G_r$ ,  $I_0 = G_{s-1}$ , and  $I_1 = G_s$  has the effect of increasing the distance between the origins in the Gibonacci digraph.

### More Intriguing Determinants

So far, the matrix entries for our determinants have had increasing indices across rows and columns. This property has facilitated the construction of representative digraphs on which the consideration of nonintersecting  $n$ -routes is manageable. Next, we wish to consider a Fibonacci matrix that does not satisfy the increasing index property. The representative digraph increases in complexity and our strategy to pair intersecting  $n$ -routes of opposite signs, expands to pair nonintersecting  $n$ -routes of opposite signs whenever possible.

The following identity first appeared as a problem in the *Fibonacci Quarterly* [11].

$$\det(A) = \det \begin{bmatrix} F_{m+3} & F_{m+2} & F_{m+1} & F_m \\ F_{m+2} & F_{m+3} & F_m & F_{m+1} \\ F_{m+1} & F_m & F_{m+3} & F_{m+2} \\ F_m & F_{m+1} & F_{m+2} & F_{m+3} \end{bmatrix} = F_{2m} F_{2m+6}. \quad (16)$$

To combinatorially see the determinant, consider the digraph given in Figure 14.

[width = 4.0in]figure14

Figure 14: This digraph is used to prove equation (16). For convenience, unlabeled vertices will be referred to by position. So  $v_{s,r}$  is the vertex in column  $s$  on level  $r$ .

A path from  $o_1$  or  $o_4$  to any destination remains at the same level (call it *level one* or *level four*) until its final (vertical) arc. The number of paths from these origins to a particular destination is the Fibonacci number associated with the length of the path. Paths from  $o_2$  to  $d_1$  or  $d_3$  must travel along level one. However a path from  $o_2$  to  $d_2$  may travel along level one or may remain at level two until its final arc. There are  $F_{m+1}$  level one paths and  $F_{m+2}$  level two paths from  $o_2$  to  $d_2$ , for a total of  $F_{m+3}$  paths. The number of paths from  $o_2$  to  $d_4$ ,  $o_3$  to  $d_1$ , and  $o_3$  to  $d_3$  can be found similarly.

Counting nonintersecting 4-routes in Figure 14 remains a daunting task. To simplify it, we classify nonintersecting 4-routes according to the number of levels used and then pair off some of those having opposite sign.

Table 1 counts the nonintersecting 4-routes using exactly two levels, namely level one and level four. Notice that the permutations associated with these eight 4-routes depend on the parity of  $m$ , but either way, the total signed sum remains the same at  $-4$ .

$m$ odd		$m$ even	
Permutation	Signed number of 4-routes	Permutation	Signed number of 4-routes
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_4 & d_3 & d_2 & d_1 \end{pmatrix}$	+1	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_4 & d_1 & d_2 \end{pmatrix}$	+1
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_4 & d_2 & d_3 & d_1 \end{pmatrix}$	-1	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_2 & d_4 & d_1 & d_3 \end{pmatrix}$	-1
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_4 & d_1 & d_2 & d_3 \end{pmatrix}$	-2	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_4 & d_3 & d_2 \end{pmatrix}$	-2
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_2 & d_3 & d_4 & d_1 \end{pmatrix}$	-2	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_2 & d_1 & d_4 \end{pmatrix}$	-2
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_2 & d_1 & d_4 & d_3 \end{pmatrix}$	+1	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$	+1
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_3 & d_2 & d_4 \end{pmatrix}$	-1	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_1 & d_4 & d_2 \end{pmatrix}$	-1
Total	-4	Total	-4

Table 1: Counting nonintersecting 4-routes restricted to levels one and four in Figure 14 .

Next we consider the 4-routes using three or more levels. Let  $P = (P_1, P_2, P_3, P_4)$  be a 4-route where  $\sigma$  is the associated permutation and  $P_i$  is the path from  $o_i$  to  $d_{\sigma(i)}$  ( $i = 1, \dots, 4$ ). We show that the signed sum of 4-routes having  $|\sigma(1) - \sigma(2)| = 2$  is zero. If  $|\sigma(1) - \sigma(2)| = 2$  and  $P_2$  is a level two path, moving  $P_1$  to level two and  $P_2$  to level one exchanges the destinations of  $o_1$  and  $o_2$ , producing a 4-route of opposite sign. So  $P' = (P_2, P_1, P_3, P_4)$  is a 4-route associated to  $\sigma' = \sigma(12)$ . If  $|\sigma(1) - \sigma(2)| = 2$  and  $P_2$  is not a level two path, then  $P_3$  must be a level three path. Moving  $P_3$  to level four and  $P_4$  to level three exchanges the destinations of  $o_3$  and  $o_4$ , producing a 4-route of opposite sign. On the set of 4-routes using three or more levels with the restriction that  $|\sigma(1) - \sigma(2)| = 2$ , this exchange produces a bijection between the even and odd members. Thus it remains to consider only those 4-routes on three or more levels where  $|\sigma(1) - \sigma(2)| = 1$  or 3.

Table 2 counts the nonintersecting 4-routes using all four levels where  $|\sigma(1) - \sigma(2)|$  is odd. Notice that the signed sum is independent of the parity of  $m$ .

Permutation	Signed number of 4-routes	Configurations
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$	$+F_{m+2}^2 F_{m+3}^2$	[width=1.5in]table2image1
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_4 & d_3 & d_2 \end{pmatrix}$	$-F_m F_{m+1} F_{m+2} F_{m+3}$	[width=1.5in]table2image2
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_2 & d_1 & d_4 \end{pmatrix}$	$-F_m F_{m+1} F_{m+2} F_{m+3}$	[width=1.5in]table2image3
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_4 & d_1 & d_2 \end{pmatrix}$	$+F_m^2 F_{m+1}^2$	[width=1.5in]table2image4
Total	$(F_{m+2} F_{m+3} - F_m F_{m+1})^2$	

Table 2: Counting nonintersecting 4-routes using all four levels in Figure 14 where the destination indices of  $o_1$  and  $o_2$  differ by an odd number.

The next step is to pair off the remaining 4-routes, which use exactly three levels and satisfy  $|\sigma(1) - \sigma(2)| = 1$  or 3. We present the details for 4-routes using level two—the symmetry of the digraph guarantees that the argument using level three is similar. Suppose  $\mathcal{P}$  is the set of nonintersecting 4-routes using levels one, two, and four with  $|\sigma(1) - \sigma(2)|$  odd. If we partition  $\mathcal{P}$  into subsets  $\mathcal{P}_{ij}$  where  $\sigma(1) = i$  and  $\sigma(2) = j$ , we find that  $\mathcal{P} = \mathcal{P}_{12} \cup \mathcal{P}_{14} \cup \mathcal{P}_{32} \cup \mathcal{P}_{34}$ . The three correspondences below completely match the even and odd elements of  $\mathcal{P}$ .

1.  $\mathcal{P}_{34}$  is in one-to-one correspondence with members of  $\mathcal{P}_{14}$  where the level one path ends with two steps.

Given a 4-route in  $\mathcal{P}_{34}$ , exchange the roles of  $d_1$  and  $d_3$  to create a 4-route in  $\mathcal{P}_{14}$ . Add two steps  $(v_{m,1}, v_{m+1,1})$  and  $(v_{m+1,1}, v_{m+2,1})$ . Extend the path that previously terminated at  $d_1$  by the jump  $(v_{m+2,4}, v_{m,4})$ . Move the terminal vertical arcs appropriately. The associated permutations differ by the transposition (13) and so they are of opposite sign. See Figure 15.

[width = 4.0in]figure15

Figure 15: Illustrating the bijections between  $\mathcal{P}_{34}$  and members of  $\mathcal{P}_{14}$  where the level one path ends with two steps.

2.  $\mathcal{P}_{32}$  is in one-to-one correspondence with members of  $\mathcal{P}_{12}$  where the level one path uses the vertex  $v_{m,1}$ .

Given a 4-route in  $\mathcal{P}_{32}$ , exchange the roles of  $d_1$  and  $d_3$  to create a 4-route in  $\mathcal{P}_{12}$ . On level four, take the arcs in the length two path from  $v_{m+1,4}$  to  $v_{m-1,4}$  and use these arcs to extend the path from  $v_{m,1}$  to  $v_{m+2,1}$ . Add the jumps  $(v_{m+2,4}, v_{m,4})$  and  $(v_{m+1,4}, v_{m-1,4})$ . Move the terminal vertical arcs appropriately. See Figure 16.

[width = 4.0in]figure16

Figure 16: Illustrating the bijections between  $\mathcal{P}_{32}$  and members of  $\mathcal{P}_{12}$  where the level one path uses the vertex  $v_{m,1}$ .

3. The members of  $\mathcal{P}_{14}$  where the level one path does not end with two steps are in one-to-one correspondence with the members of  $\mathcal{P}_{12}$  where the level one path does not pass through vertex  $v_{m,1}$ .

Given a 4-route in  $\mathcal{P}_{14}$  that does not end in two steps, exchange the roles of  $d_2$  and  $d_4$  to create a 4-route in  $\mathcal{P}_{12}$ . The path on level one has the form  $Q(v_{m,1}, v_{m+2,1})$  or  $Q'(v_{m-1,1}, v_{m+1,1})(v_{m+1,1}, v_{m+2,1})$  where  $Q$  is a path of length  $m$  and  $Q'$  is a path of length  $m-1$ . Level two contains a path of length  $m-1$ , call it  $R$ . Move  $R$  to level one and insert the jump  $(v_{m-1,1}, v_{m+1,1})$  and the step  $(v_{m+1,1}, v_{m+2,1})$ . The path on level two becomes either  $Q(v_{m+1,2}, v_{m+2,2})$  or  $Q'(v_{m,2}, v_{m+2,2})$  depending on the form of the original path on level one. Finally, add a jump  $(v_{m+1,4}, v_{m-1,4})$ . Move the terminal vertical arcs appropriately. See Figure 17.

[width = 4.0in]figure17

Figure 17: Illustrating the bijections between members of  $\mathcal{P}_{14}$  where the level one path does not end and members of  $\mathcal{P}_{12}$  where the level one path does not pass through vertex  $v_{m,1}$ .

Each correspondence is reversible and exchanges two destinations, so the permutations associated to the paired 4-routes have opposite sign and their signed sum will be zero. The only 4-routes to contribute to the determinant are those counted in Tables 1 and 2. Thus  $\det(A) = (F_{m+2}F_{m+3} - F_mF_{m+1})^2 - 4$ . Identities 15 and 88 from [2] can be used to simplify the result to  $(F_{m+2}F_{m+3} - F_mF_{m+1})^2 - 4 = F_{2m+3}^2 - F_3^2 = F_{2m}F_{2m+6}$ .

We can extend the determinant in (16) to have Gibonacci entries. The modified digraph weights the arcs from each origin vertex as shown in Figure 18.

[width = 4.5in]figure18

Figure 18: This digraph is used to prove equation (17). The arcs exiting each origin have weight  $G_0$  or  $G_1$  as indicated above.

Using an argument similar to the proof of equation (7), each nonintersecting 4-route given in Table 1 contributes  $(G_0G_2 - G_1^2)^2$  towards the signed sum of the weighted 4-routes of the modified graph. Using three or more levels, the same bijections match 4-routes of equal weight but opposite sign. Using exactly three levels when  $|\sigma(1) - \sigma(2)|$  is odd changes the entries in Table 2 from Fibonacci numbers to Gibonacci numbers. We conclude that

$$\det \begin{bmatrix} G_{m+3} & G_{m+2} & G_{m+1} & G_m \\ G_{m+2} & G_{m+3} & G_m & G_{m+1} \\ G_{m+1} & G_m & G_{m+3} & G_{m+2} \\ G_m & G_{m+1} & G_{m+2} & G_{m+3} \end{bmatrix} = (G_{m+2}G_{m+3} - G_mG_{m+1})^2 - 4(G_0G_2 - G_1^2)^2. \quad (17)$$

While every determinant can be represented by some weighted digraph, the focus of this paper was coming up with particular presentations that made the computations more intuitive and generalizable. Further work remains designing useful digraphs to discover and understand other identities.

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